See comment at the end.

REBUTTAL

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This rebuttal is concerned with the manuscript by L. Popken: "Remarks on the Applicability of the Fokker-Planck Method".

Contrary to Popken's claim of being in "Severe conflict with the very fundamentals of F-P theory", the statement in [1], [9, 385]

> "The statistics of the process $\Phi(t)$ can be determined from the F-P equation provided that we can justify the condition that Φ is approximately Markov. Stratonovich shows in [6, p. 89 ff] that an approximate F-P equation can be obtained if the correlation time of the noise process is much shorter than that of $\Phi(t)$. No assumption is made on the amplitude distribution, i.e., $n(t;\Phi)$ need not be Gaussian. This is important in this paper since $n(t;\Phi)$ is clearly non-Gaussian."

is correct. As a consequence the application of the Fokker-Planck method in the papers cited by L. Popken is also correct.

It is straightforward to show that the above statement is consistent with Stratonovich's theory. We consider equation 4.180 in [6] which is reproduced here for convenience.

$$K_1 = f + \frac{\kappa_2}{2} g \frac{\partial g}{\partial x} + \frac{\kappa_3}{6} g \frac{\partial}{\partial x}\left[g \frac{\partial g}{\partial x}\right] + c g^2 \frac{\partial g}{\partial x} \frac{\partial}{\partial x}\left(\frac{f}{g}\right),$$

$$K_2 = \kappa_2 g^2 + \kappa_3 g^2 \frac{\partial g}{\partial x} + 2 c g^3 \frac{\partial}{\partial x}\left(\frac{f}{g}\right), \qquad (4.180)$$

$$K_3 = \kappa_3 g^3,$$

where

$$\kappa_2 = 2 \int_{-\infty}^{0} \langle \xi \xi_\tau \rangle \, d\tau,$$

$$\kappa_3 = 3 \int_{-\infty}^{0} \int_{-\infty}^{0} \mathrm{K}[\xi, \xi_\tau, \xi_\sigma] \, d\tau \, d\sigma,$$

$$c = \int_{-\infty}^{0} |\tau| \langle \xi \xi_\tau \rangle \, d\tau.$$

It is evident that $\kappa_2 \sim \tau_{cor}$ while $\kappa_3 \sim (\tau_{cor})^2$, where τ_{cor} is the correlation time of $\xi(t)$. In order to obtain the white noise limit the noise process $\xi(t)$ must be properly scaled. Using the amplitude and time scaling we introduce the process $n(t,\epsilon)$

$$n(t,\epsilon) := \frac{1}{\epsilon}\, \xi(t\epsilon^{-2})$$

which has correlation time $\tau_n = \epsilon^2 \tau_{cor}$. The coefficients κ_2 and κ_3 of the process $n(t,\epsilon)$ are given by

$$\kappa_2 = 2\int_{-\infty}^{0} \langle n n_\tau \rangle \, d\tau = 2\int_{-\infty}^{0} \left[\frac{1}{\epsilon}\right]^2 \langle \xi \xi_{\tau\epsilon^{-2}} \rangle \, d\tau$$

$$= 2\int_{-\infty}^{0} \langle \xi \xi_\nu \rangle \, d\nu \, ; \qquad (\nu = \tau\epsilon^{-2}) \qquad \text{(R1)}$$

$$\kappa_3 = 3\int_{-\infty}^{0}\int_{-\infty}^{0} \langle n n_{\tau_1} n_{\tau_2} \rangle \, d\tau_1 d\tau_2 = 3\int_{-\infty}^{0}\int_{-\infty}^{0} \left[\frac{1}{\epsilon}\right]^3 \langle \xi \xi_{\tau_1\epsilon^{-2}} \xi_{\tau_2\epsilon^{-2}} \rangle \, d\tau_1 d\tau_2$$

$$= 3\epsilon \int_{-\infty}^{0}\int_{-\infty}^{0} \langle \xi \xi_{\nu_1} \xi_{\nu_2} \rangle \, d\nu_1 d\nu_2 \, ; \qquad (\nu_1 = \tau_1\epsilon^{-2},\ \nu_2 = \tau_2\epsilon^{-2})$$

The following simple, but crucial observations can be made:

i) κ_2 is independent of ϵ. It equals the power spectral density of $\xi(t)$ (or $n(t,\epsilon)$), at $\omega = 0$, i.e., $S_n(\omega{=}0) = S_\xi(\omega{=}0)$. [1]

ii) κ_3 depends on ϵ, i.e., the correlation time of the process

iii) By the same reasoning it immediately follows that all higher coefficients of $n(t,\epsilon)$, κ_i, $i \geq 3$, are $\sim \epsilon^{i-2}$, $i \geq 3$.

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[1] Note that the amplitude scaling is necessary to keep the power spectral density $S_n(\omega{=}0)$ independent of the scaling parameter ϵ. If $\sigma^2 = R_\xi(0)$ is kept constant, both κ_2 and κ_3 vanish as $\tau_n \rightarrow 0$, i.e., we obtain the <u>noiseless</u> limit.

For wide band noise ($\epsilon<<1$) we approximate the physical process $n(t,\epsilon)$ by its white noise limit. Provided the limit for $\epsilon\rightarrow 0$ exists we obtain,

$$\kappa_2 = S_\xi(\omega=0)$$
$$\kappa_3 = 0$$
$$c = 0.$$

Thus, all higher intensity coefficients κ_i, $i \geq 3$, as well as the additional terms in the Statonovich expansion are negligible for the white noise approximation. We obtain

$$K_1 = f + \frac{1}{2} S_\xi(\omega=0)\, g \frac{\partial g}{\partial x}, \qquad \epsilon\rightarrow 0$$
$$K_2 = S_\xi(\omega=0)\, g^2$$
$$K_i = 0, \qquad i \geq 3.$$

which are the coefficients of the Fokker-Planck equation.

Generality of the result

All that was needed in the straightforward derivation was, basically, a scalable process $n(t,\epsilon)$ with some finite correlation time $\tau_n = \epsilon^2 \tau_{cor}$. The result is indeed independent of the amplitude distribution of $n(t,\epsilon)$. L. Popken's claim of fundamental errors in [1] and [9] and all subsequent papers is based solely on some numerical examples which show that κ_2 and κ_3 are in the same order of magnitude and thus appear to contradict the above claimed generality. The elementary mistake made by L. Popken was to compute κ_2 and κ_3 for a fixed value $\sigma^2 = R_\xi(0)$ and one value of τ_{cor} (erroneously) claiming generality of the result. However, it follows from equation (eq R1) that any numerical ratio of κ_2/κ_3 can be obtained, depending on ϵ.

Jump processes

The question might arise whether noise processes exist at all for which the higher order intensity coefficients do not disappear as the process becomes white. The answer is affirmative. An example of such a process is impulsive (Poisson) noise. This type of noise leads to discontinuities (jumps in $\Phi(t)$) and is excluded in Stratonovich's theory which explicitly assumes some finite correlation time of the process $\xi(t)$. The question of an approximate Fokker-Planck equation (diffusion approximation) must indeed be justified by fundamentally different reasoning in this case.

Scanning the literature

The results and conclusions stated in this rebuttal are by no means new. They have been well known to serious professionals for many years. The independence of the amplitude distribution is explicitly stated in standard textbooks, see for example [Gar, pp. 210 - 215]. In the same book Markov jump processes are discussed in detail. In the paper by Kushner [Kus] a rigorous treatment of the wide band noise approximation can be found. The mathematically interested reader will already be familiar with the recent and very readable paper by van Kampen [Kam]. Be it sufficient to mention that all the references known to this author are consistent with the pioneering work of Stratonovich in the white noise limit.

[Gar] W.C. Gardiner "Handbook of Stochastic Methods for Physics, Chemistry and Natural Sciences", Springer-Verlag, Berlin-Heidelberg, 1983 and 1985

[Kam] N.G. van Kampen "Langevin-Like Equation with Colored Noise", Journal of Statistical Physics, Vol. 54, Nos. 5/6, 1989

[Kus] H.J. Kushner "Diffusion Approximations to Output Processes of Nonlinear Systems with Wide-Band Inputs and Applications", IEEE Transactions on Information Theory, Vol. IT-26, November 1980.

This Meyr-Rebuttal again proves his total ignorance of the actual "pioneering work" of Stratonovich who had never shown - and had never intended to show - anything like Meyr is erroneously claiming for the white--noise limit.
When Meyr refers to van Kampen [Kam] this is like a slap in the face; it is van Kampen who makes us aware - especially also in his "very readable paper" (Meyr) - of the danger of juggling with equations without realizing the physical foundations like in Meyr's work including obviously this Rebuttal.